

Comment on “Spherical $2 + p$ spin-glass model: An analytically solvable model with a glass-to-glass transition”

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(Dated: February 1, 2008)

Guided by old results on simple mode-coupling models displaying glass-glass transitions, we demonstrate, through a crude analysis of the solution with one step of replica symmetry breaking (1RSB) derived by Crisanti and Leuzzi for the spherical $s + p$ mean-field spin glass [Phys. Rev. B **73**, 014412 (2006)], that the phase behavior of these systems is not yet fully understood when s and p are well separated. First, there seems to be a possibility of glass-glass transition scenarios in these systems. Second, we find clear indications that the 1RSB solution cannot be correct in the full glassy phase. Therefore, while the proposed analysis is clearly naive and probably inexact, it definitely calls for a reassessment of the physics of these systems, with the promise of potentially interesting new developments in the theory of disordered and complex systems.

In a recent paper¹, Crisanti and Leuzzi have proposed a detailed analysis of the so-called spherical $s + p$ mean-field spin-glass model, defined by the Hamiltonian

$$\mathcal{H} = \sum_{i_1 < \dots < i_s}^{1,N} J_{i_1 \dots i_s}^{(s)} \sigma_{i_1} \dots \sigma_{i_s} + \sum_{i_1 < \dots < i_p}^{1,N} J_{i_1 \dots i_p}^{(p)} \sigma_{i_1} \dots \sigma_{i_p},$$

where s and p are integers such that $2 \leq s < p$, the spins σ_i are N real variables subject to the spherical constraint $\sum_{i=1}^N \sigma_i^2 = N$, and the random coupling constants $J_{i_1 \dots i_s}^{(s)}$ and $J_{i_1 \dots i_p}^{(p)}$ are uncorrelated zero mean Gaussian variables with variances $s!J_s^2/2N^{s-1}$ and $p!J_p^2/2N^{p-1}$, respectively. At the inverse temperature β , these variances enter into the definition of the control parameters of the model, $\mu_s = (\beta J_s^2)^2 s/2$ and $\mu_p = (\beta J_p^2)^2 p/2$.

In fact, Ref. 1 mostly deals with the case $s = 2$, $p \geq 4$, which displays a rich phase diagram with four different phases and a variety of transitions between them². The case $2 < s < p$, to which the present Comment is devoted, is only discussed shortly in an appendix and illustrated with the $3 + 4$ model, whose static and dynamical phase diagrams both exhibit only two phases, paramagnetic with replica symmetry (RS) and glassy with one step of replica symmetry breaking (1RSB), separated by smooth transition lines. This result is claimed to be generic.

In this Comment, we show that this might not be the case and that there could be a parameter domain apparently left unexplored by all previous workers and corresponding to $2 < s < p$ and $p - s$ large enough, where new phenomena occur. To achieve this goal, we first recall seemingly little known results about simple schematic models of the mode-coupling theory (MCT) for the liquid-glass transition^{3,4,5}, which show glass-glass (G-G) transitions and higher-order singularities. Then, guided by the insight gained within the framework of the MCT, we report the results of a naive investigation of the 1RSB solution derived by Crisanti and Leuzzi, which raise a number of issues about the structure of the phase diagram of the systems with large $p - s$.

In the framework of the MCT^{3,4,5}, the schematic models are minimal models reproducing the typical nonlin-

earities and bifurcation scenarios of the dynamical equations generically derived within this theory. A particularly important class consists of the so-called F_{pq} models, $1 \leq p < q$, defined by the memory kernels $m(t) = v_p \phi(t)^p + v_q \phi(t)^q$. They have been widely studied and, although most of the works have dealt with the F_{12} and F_{13} models³, scattered results are also available on more general F_{pq} models^{6,7}.

The bifurcation analysis of the solutions of the F_{pq} equation with $1 < p < q$ leads to a parametric representation of the dynamical transition line, which uses the non-ergodicity parameter $f = \lim_{t \rightarrow +\infty} \phi(t)$ as the variable. It reads

$$v_p(f) = \frac{1}{q-p} \frac{q-1-qp}{f^{p-1}(1-f)^2}, \quad v_q(f) = \frac{1}{p-q} \frac{p-1-pf}{f^{q-1}(1-f)^2},$$

with $(p-1)/p \leq f \leq (q-1)/q$. Topological changes occur in this curve depending on p and q , or more precisely on their combination⁸

$$\Lambda(p, q) = \frac{\sqrt{pq} - \sqrt{(p-1)(q-1)}}{\sqrt{2}}.$$

If p and q are close enough, such that $\Lambda(p, q) < 1$, the curve $\{v_p(f), v_q(f)\}$ is smooth, but, if they are widely separated, such that $\Lambda(p, q) > 1$, a loop appears in the glassy domain. Using then a dynamical stability criterion which states that, when several values of the non-ergodicity parameter seem possible at a given point (v_p, v_q) , only the largest one is physically meaningful³, one shows that two of the three branches of the loop have to be discarded, while the remaining one forms a G-G transition line which terminates at an A_3 higher-order singularity (the ordinary MCT bifurcation is of type A_2)³. In the marginal cases like the models with $p = 2$ and $q = 9$ or $p = 9$ and $q = 50$, where $\Lambda(p, q) = 1$, the loop reduces to a point and yields an A_4 singularity. These results are illustrated in Fig. 1 for different models with $p = 2$. Interestingly, exactly the same dynamical scenarios are obtained in MCT studies of colloidal suspensions with short-ranged attractions^{9,10}.

Let's now come back to the original spin-glass problem. One of the motivations for the study of the mean-field

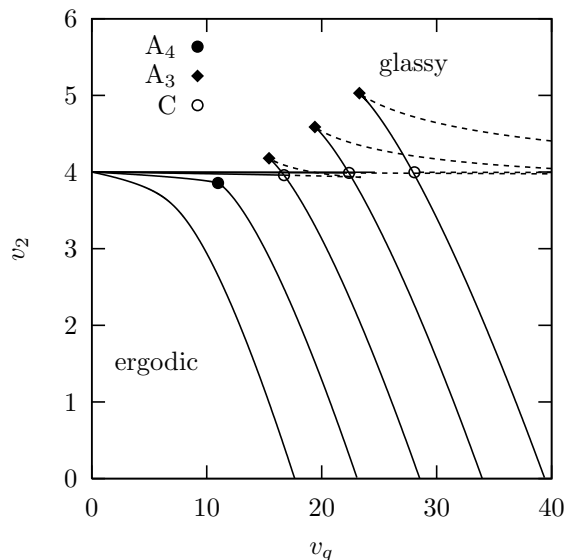


FIG. 1: Phase diagrams of the F_{pq} schematic models with $p = 2$ and $q = 7, 9, 11, 13$, and 15 (from left to right). The solid lines are the true transition lines and the dashed lines are the portions of the $\{v_p(f), v_q(f)\}$ parametric curves which are excluded by the dynamical stability criterion. The higher-order singularities (A_3 and A_4) and the crossing points (C) delimiting the G-G transition lines are indicated by symbols.

spherical spin-glass models with multispin interactions has been the finding that the time evolution of the spin correlation function in the Langevin dynamics of these systems^{11,12,13,14} is described at high temperature by schematic mode-coupling equations. Namely, the high-temperature dynamics of the $s + p$ spin glass is described by the $F_{(s-1)(p-1)}$ schematic model, with the replacement $v_{s-1} = \mu_s$ and $v_{p-1} = \mu_p$. This analogy turned out to be a fruitful source of new theoretical developments in the physics of structural glasses, allowing to complement the MCT, originally a purely dynamical theory derived under the assumption of equilibrium, with very interesting static¹⁵ and aging¹⁶ scenarios. So, since the F_{pq} schematic models with well separated indices show interesting and nontrivial bifurcation scenarios in their glassy domain, one might legitimately wonder whether similar features could be present in the glassy phase of the $s + p$ spin glass as studied by Crisanti and Leuzzi¹.

We first consider the statics. Its method of solution within the 1RSB scheme is described in detail in Ref. 1 and we only recall the expression of the free energy per spin Φ in the 1RSB phase as a function of the overlap parameter q and of the usual 1RSB parameter m . It reads $-\beta\Phi(q, m) = s_\infty + G(q, m)$, where s_∞ is the entropy per spin at infinite temperature and $G(q, m)$ is given by

$$2G(q, m) = \frac{\mu_s}{s} + \frac{\mu_p}{p} + (m-1) \left(\frac{\mu_s}{s} q^s + \frac{\mu_p}{p} q^p \right) + \frac{1}{m} \ln(1 - q + mq) + \frac{m-1}{m} \ln(1 - q).$$

In this expression, the coupling constants μ_s and μ_p are functions of q and m as well, obtained from the stationarity conditions $\partial_q G(q, m) = 0$ and $\partial_m G(q, m) = 0$.

In the same spirit as in the schematic MCT calculation, one is led to consider the parametric surface $\{\mu_s(q, m), \mu_p(q, m), G(q, m)\}$, which provides one with an implicit representation of the free energy as a function of the control parameters, and to look for topological changes when s and p are varied. As illustrated for the $3 + 16$ model in the insert of Fig. 2, where a part of this surface is plotted, such changes indeed occur when $p - s$ is increased. A swallowtail, where the free energy as a function of μ_3 and μ_{16} would be multivalued, is clearly visible above the surface. This looks very much like the loops met in the van der Waals theory of phase coexistence. Following this analogy, since the saddle-point calculation of the free energy requires that G should be minimized for given μ_s and μ_p , a crude way of dealing with this feature of the surface would be to discard it in order to define uniquely the free energy at a given state point. Then, this leaves one with a line of double points (a “binodal”), along which two distinct (q, m) pairs correspond to the same value of the free energy of the glassy phase. When this line is crossed, by following a path of constant μ_3 for instance, a discontinuity in (q, m) occurs, while the free energy remains continuous. At the level of the present discussion, this line appears as a candidate for a line of static G-G transitions, starting at a “triple” point, where the paramagnetic phase and two 1RSB phases with $m = 1$ and different values of q would have the same free energy, and ending at a “critical” point, where the doublet (q, m) is such that $\partial_q \mu_s \partial_m \mu_p - \partial_m \mu_s \partial_q \mu_p = 0$ and $\partial_q (\partial_q \mu_s \partial_m \mu_p - \partial_m \mu_s \partial_q \mu_p) = 0$ (the first equality characterizes “spinodals” corresponding to the edges of the swallowtail and meeting at the “critical” point).

The resulting static phase diagram for the $3 + 16$ model would then be the one outlined in Fig. 2. Its shape seems generic for well separated s and p , for instance, $s = 3$ and $p \geq 13$, $s = 4$ and $p \geq 23$, or $s = 10$ and $p \geq 105$. The larger s , the larger $p - s$ has to be for a candidate static G-G transition line to exist.

We now turn to the 1RSB solution for the dynamics of the $s + p$ model. Following Crisanti and Leuzzi, the discussion is not based on a direct analysis of the off-equilibrium dynamics of the system, but on the study of its complexity Σ , a procedure which should yield identical results. Thus it amounts here again to the consideration of a set of parametric equations, where μ_s , μ_p , and Σ are expressed as functions of the dynamical overlap and 1RSB parameters. See Ref. 1 for the corresponding expressions.

Like in the static calculation, the parametric surface $\{\mu_s(q, m), \mu_p(q, m), \Sigma(q, m)\}$ becomes singular for large $p - s$, with domains where $\Sigma(\mu_s, \mu_p)$ would be multivalued. This is illustrated in Fig. 3 for the $3 + 16$ model, where one can also see that the surface is more complicated than in the statics (at this point, it greatly helps to inspect the surface with an interactive computer graphic

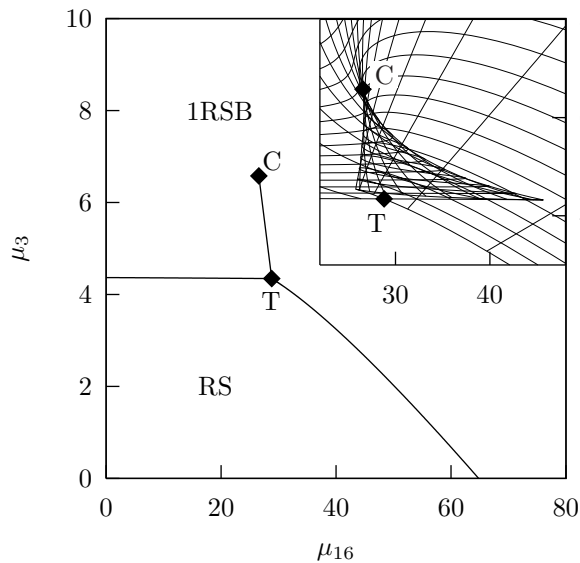


FIG. 2: Putative 1RSB static phase diagram of the 3 + 16 model in the (μ_3, μ_{16}) plane. T and C denote the “triple” and “critical” points and delimit the candidate G-G transition line. Insert: Projection onto the (μ_3, μ_{16}) plane of the parametric surface $\{\mu_3(q, m), \mu_{16}(q, m), G(q, m)\}$ in the region of the candidate G-G transition line. The outline of a swallowtail is clearly visible.

software in order to fully appreciate the nature of the singularities). Following again the analogy with the van der Waals theory, since the calculation of the dynamical solution requires that Σ should be maximized, a naive way to define a unique complexity at a given state point (μ_s, μ_p) is to discard the parts of the parametric surface which do not correspond to the largest possible value of Σ . In the case of the 3 + 16 model, this reduction leads to the appearance of three distinct curve portions, successively delimited by points A, B, C, and D, in Fig. 3, which, when crossed, for instance along lines of constant μ_3 , are associated with discontinuities in (q, m) .

Line AB is a portion of the dynamical $m = 1$ line¹⁷. Indeed, in the corresponding domain of μ_3 and μ_{16} , the sheet of the parametric surface originating in the plane $\mu_3 = 0$ (the right part of the surface in Fig. 3) overhangs the one originating in the plane $\mu_{16} = 0$ (the left part). This means in particular that, when point A is approached along the border between the paramagnetic and glassy phases, the complexity obtained when coming from the large q side (from the right) is strictly larger than the one obtained when coming from the small q side (from the left). Thus, when the representative point of the system crosses the portion of the $m = 1$ line comprised between points A and B, discontinuities in (q, m) (with m jumping to or from unity depending of the direction of approach) and Σ occur, as the result of jumping from one sheet of the complexity surface to the other. At point B, the two sheets intersect for the first time. The $m = 1$ line continues below the reduced complexity sur-

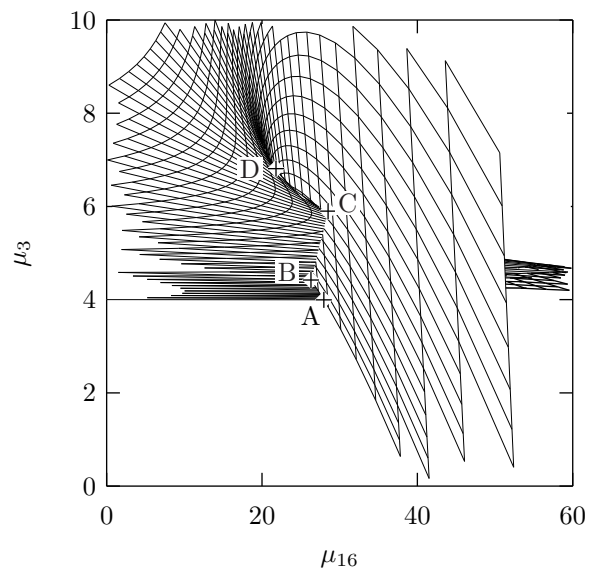


FIG. 3: Top view of the parametric surface $\{\mu_3(q, m), \mu_{16}(q, m), \Sigma(q, m)\}$ for the dynamical 1RSB phase of the 3 + 16 model. Except in the rightmost part of the figure where the surface is cut off in order to show an underlying swallowtail, the lower parts of the surface are hidden, so that only the point corresponding to the largest complexity is visible for a given (μ_3, μ_{16}) pair. Points A, B, C, and D, marked by crosses on a white background, delimit the different branches of the putative dynamical G-G transition line (see text for details).

face and becomes irrelevant (at least at the level of the present crude analysis). Point B marks the beginning of a line of double points, line BC, along which two distinct (q, m) pairs correspond to the same value of the complexity of the glassy phase. When this line is crossed, (q, m) is discontinuous, while Σ is continuous. Eventually, at point C, the line of double points terminates at a line of cusp singularities which extends towards the “critical” point D, providing a final line of (q, m) discontinuities, with a discontinuous Σ anew.

As already suggested about the statics, these lines of (q, m) discontinuities appear as candidates for dynamical G-G transition lines. A simple condition on s and p for their existence is obtained by looking for stationary points in the dynamical $m = 1$ line, as they are necessary for the presence of a point like point A. Since the equations of this $m = 1$ line and of the transition line of the $F_{(s-1)(p-1)}$ schematic model coincide, this condition is just the same as in the MCT calculation, i.e., $\Lambda(s-1, p-1) > 1$, with Λ defined above. Thus, candidate dynamical G-G transition lines are expected in the systems with $s = 3$ and $p \geq 11$, $s = 4$ and $p \geq 16$, or $s = 10$ and $p \geq 52$, for instance. Generically, keeping s fixed and increasing p , the possibility of a G-G transition line first appears in the dynamics, then in the statics, so that there is a whole class of systems like the 3+11 model which only display a candidate dynamical G-G transition

line. Finally, there are still limiting cases like the $3 + 10$ and $10 + 51$ models, where point A itself is a stationary point of the dynamical $m = 1$ line, to which the putative dynamical G-G transition line reduces.

So far, we have only considered the topological features of the complexity surface in the dynamical 1RSB solution and not the actual values of this function. It came as a surprise in the course of the present investigation that negative values of the complexity could be found in a large parameter domain. For a given s , this phenomenon is observed for p larger than the one needed for a singularity to appear in the complexity surface, so that it does not occur for all systems with a candidate dynamical G-G transition line. Considering again the $3 + 16$ model, it is easily evidenced by computing $\Sigma(q, m)$ along lines of constant m , including $m = 1$ ¹⁷. It results that there is a whole region above the line joining points C and D in Fig. 3 and corresponding to rather small values of m , where the only available value of the 1RSB complexity is negative (this is not the case on the $m = 1$ line which falls in the domain where $\Sigma(\mu_s, \mu_p)$ is multivalued and where, for a given (μ_s, μ_p) pair, there is the possibility of finding a positive value of Σ on another sheet of the complexity surface). As pointed out by Crisanti and Leuzzi¹⁷, this finding signals an unanticipated breakdown of the 1RSB scheme and calls for a complete reexamination of the phase behavior of the systems with large $p - s$.

Note that many of the above features of the dynamics of the $s + p$ spin-glass models could have been anticipated from the work of Caiazzo *et al.*¹⁸, who studied the off-equilibrium dynamics of a lattice gas generalization of the $s + p$ spin-glass model. Indeed, for this richer and more complex system, these authors found the possibility of dynamical G-G transition lines and, along these lines, of dynamical singularities marking qualitative changes in the aging behavior of the system. They also mention

that, deeper in the glassy phase, the “one-step replica symmetry breaking solution should not hold and, instead, a spin-glass-like aging dynamics should be found”. Apparently, and quite surprisingly, the possibility that the same could be true in the simpler $s + p$ spin-glass model was not envisaged.

In conclusion, guided by previous results on simple MCT models displaying G-G transitions and higher-order singularities^{6,7}, we have proposed a crude analysis of the 1RSB solution derived by Crisanti and Leuzzi in Ref. 1 which shows that the phase behavior of the spherical $s + p$ mean-field spin glasses with well separated s and p does not appear to be fully understood yet. First, there seems to be a possibility of G-G transition scenarios both in the statics and the dynamics of these systems. Second, there are clear indications that the 1RSB solution cannot be correct in the full glassy phase¹⁷. Obviously, it results from the latter finding that the present analysis based on this 1RSB solution, and also on probably naive analogies with the van der Waals theory of phase coexistence, is necessarily at least partly inconsistent. However, it is sufficient to raise a number of issues about the structure of the phase diagram of the systems with large $p - s$. It is beyond the scope of this comment and definitely beyond the expertise of its author to answer these questions. So, we can only wish that the above observations will be judged interesting by experts in the theory of spin glasses and will motivate future studies of these systems, with potentially interesting new developments, for instance, in the theory of the amorphous-amorphous transitions in structural glasses.

As a non-expert in spin-glass theory, the author is indebted to A. Crisanti and L. Leuzzi for useful advices on the presentation of this work and for insightful observations on the physical implications of the reported results.

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